Circles and Parabolas

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This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

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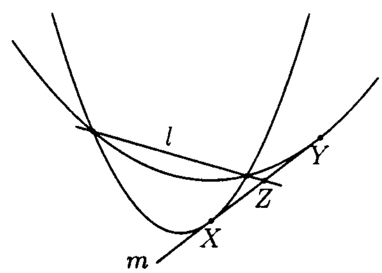
Several years ago, reading the Problems section of the *American Mathematical Monthly*, I came across the following problem:

> Consider two parabolas: $y = ax^2 + bx + c$ and $y = dx^2 + ex + f$, intersecting in two points. Let l be their common chord, and m be the tangent to both parabolas that touches them at X and Y. Then l intersects m in the point Z, which is the midpoint of XY (Figure 1).

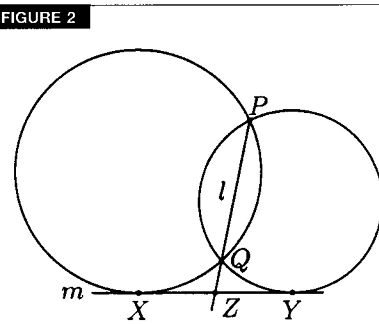
FIGURE 1



The solution used some computations in coordinates, and I started to think whether a more geometrical one exists. Then I realized that there was a similar problem about circles:

> Consider two circles intersecting in two points. Let l be their common chord, and m be the common tangent touching circles at points X and Y. Then l intersects m in the point Z that is the midpoint of XY (Figure 2).

FIGURE 2

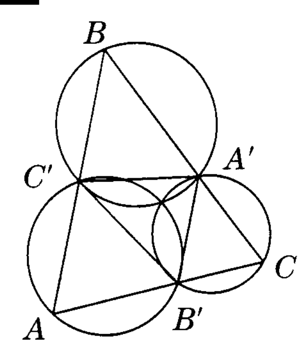


The second problem has a simple solution: it is well known that $ZX^2 = ZP \cdot ZQ$ (triangles ZPX and ZXQ are similar); for the same reasons $ZY^2 = ZP \cdot ZQ$, so $ZX = ZY$.

It turns out that there are many other examples of statements about circles parallel to statements about parabolas. Here is one:

> Let ABC be a triangle. Let point B' lie somewhere on the line AC, point C' lie somewhere on AB, and point A' lie somewhere on BC. Then the circles circumscribed around triangles $AB'C'$, $A'BC'$, and $A'B'C$ have a common intersection point (Figure 3).

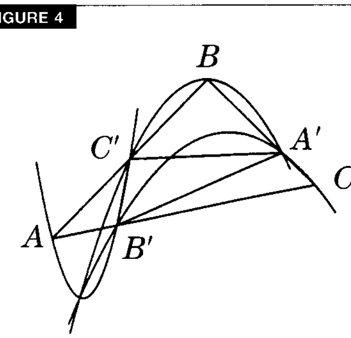
FIGURE 3



The parallel statement about parabolas reads as follows:

> Let ABC be a triangle. Let point B' lie somewhere on the line AC, point C' lie somewhere on AB, and point A' lie somewhere on BC. Then the three parabolas going through the points $AB'C'$ (the first one), $A'BC'$ (the second one), and $A'B'C$ (the third one) have a common intersection point (Figure 4).

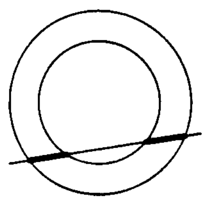
FIGURE 4



To give another example, we have to extend our dictionary giving the correspondence between notions related to circles and parabolas. For example, "concentric circles" should be translated as "parabolas with a common axis obtained one from the other by a shift along this axis". This translation is used in the following statements:

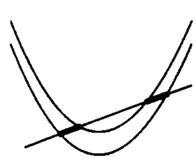
> A line intersects concectric circles at the four points A_1, B_1, B_2, A_2. Then the segments A_1B_1 and B_2A_2 are equal (Figure 5).

FIGURE 5



A line intersects "concentric" parabolas (in the sense explained above) at the four points A_1, B_1, B_2, A_2. Then the segments A_1B_1 and B_2A_2 are equal (Figure 6).

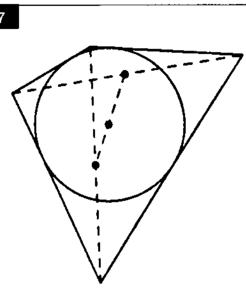
FIGURE 6



Still another extension of our dictionary would be the translation of an idiomatic expression "line goes through the center of a circle", which becomes "line is parallel to the axis of a parabola". This translation is used in the following statements:

> If a circle is inscribed in a quadrilateral, then the midpoints of the diagonals and the center of the circle lie on a straight line (Figure 7).

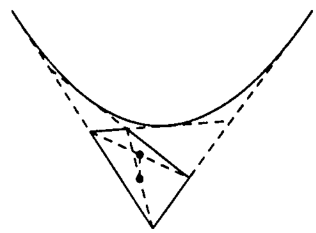
FIGURE 7



(This statement is sometimes called Newton's theorem.) The parallel statement for parabolas is:

> Four tangents to a parabola intersect to form a quadrilateral. Then the line that goes through the midpoints of the diagonals of the quadrilateral is parallel to the axis of the parabola (Figure 8).

FIGURE 8



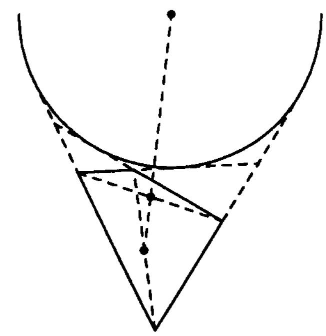
These (and many other) examples give us a strong feeling that there is some general principle saying that every true statement about circles (in a certain language) can be translated into a true statement about parabolas. Unfortunately, it is not clear how to formulate a precise general principle of this type.

Instead, let us see how the statement about parabolas can be derived from the statement about circles. Recall our first pair of statements; let us prove that the common tangent to two parabolas is divided into equal segments by their common chord (Figure 1).

Assume that it is not the case, and the intersection point is not the midpoint. A parabola can be approximated by an ellipse that has one of its focuses far away. Consider such ellipses for both parabolas. The common chord of the ellipses will be close to the common chord of the parabolas, and the common tangent for the ellipses will be close to the common tangent for the parabolas. Therefore, if ellipses are close enough to the parabolas, the common chord to them will intersect the common tangent *not* in the middle point. One may assume both ellipses to have the same ratio of axes. Then we can apply an affine transformation to transform the ellipses into two circles for which the common chord does not intersect the common tangent at the midpoint, which is impossible. Q.e.d.

A similar argument can be applied to other examples given above. However, some additional tricks are needed. For example, consider the statement about the four tangents (Figure 8). If we try to use the same method, we come to a picture that differs from Figure 7: see Figure 9.

FIGURE 9



However, if the statement (saying that the center of the circle and the midpoints of diagonals lie on a straight line) is true for Figure 7, it should be also true for Fig. 9. The explanation goes as follows. Consider polar coordinates on the circle; let ϕ_1, ϕ_2, ϕ_3, ϕ_4 be the angle coordinates of the tangent points. Then the coordinates of all other points are rational functions of $\sin\phi_1$, $\cos\phi_1$, $\sin\phi_2$, $\cos\phi_2$, etc. Using the substitution $t_i = \tan(\phi_i/2)$, we see that the coordinates of all points are rational functions of t_1, t_2, t_3, t_4. The statement in question (three points lie on a straight line) is an identity involving those coordinates. Therefore, if it is true in the neighborhood of some point (as Figure 7 shows), it should be true for all values of t_i, and therefore also for the Fig. 9 configuration.

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Letter to the Column Editor

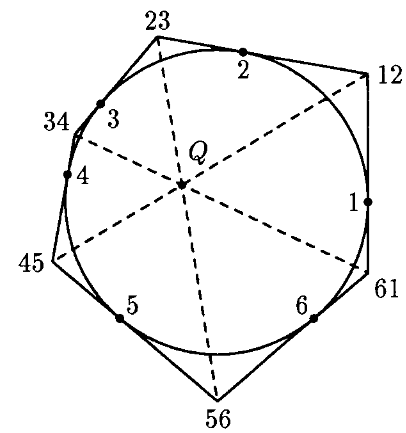
In your article on 3-dimensional proofs of planar theorems, one of my favorite examples of that kind was missing. Do you know the following 3-dimensional proof of Brianchon's Theorem (saying that the main diagonals of a hexagon circumscribed about a circle meet at one point)?

Proof: Consider the circle as a plane section going through the center of a hyperboloid of one sheet. There are two families of straight lines on the hyperboloid, say A and B, such that through every point on the hyperboloid passes exactly one line of each family, and every line in family A meets every line in family B. Let 1, 2, . . . , 6 be the six points where the sides of the hexagon are tangent to the circle. Consider the lines $l_1, \ldots, l_6$ that lie on the hyperboloid and pass through $1, \ldots, 6$, respectively. Lines l_1, l_3, l_5 are in family A and l_2, l_4, l_6 are in family B. Since the neighbor lines are in different families, they intersect to form a 3-dimensional hexagon; its top view is our original hexagon. (The vertices of this 3-dimensional hexagon are denoted by 12, 23, 34, 45, 56, 61 in the sequel.)

Consider now lines l_1 and l_4. They belong to different families, so they intersect each other and lie in some plane p_{14}. Planes p_{25} and p_{36} are defined in a similar way. Now let us look at the intersection line of planes p_{14} and p_{25}. Points 12 and 45 lie on both planes, so the intersection of these two planes is the diagonal 12–45. Since we have three planes, there are three intersection lines (12–45, 23–56, 34–61), and the point Q where the three planes meet is the point where these three lines meet. The top view of each of these lines is a main diagonal of our original plane hexagon, hence the top view of Q is the point where the main diagonals meet.

There is a version of this proof which works for all fields k of characteristic $\neq 2$ ("circle" must be replaced by "conic"; in characteristic 2 the Brianchon theorem makes no sense, since in that case all the tangent lines to a conic meet at one point and hence the diagonals of a circumscribed hexagon are not defined). Suppose that we have a conic C in a projective plane P^2 over k. We may assume that k is algebraically closed, that P^2 lies in the 3-dimensional projective space P^3 with homogeneous coordinates $(x{:}y{:}z{:}w)$, that P^2 is given by the equation $x + w = 0$, and that the conic C is the intersection of the "hyperboloid" H given by the equation $xw - yz = 0$ with P^2. Writing every point $(x{:}y{:}z{:}w) \in P^3$ as a matrix with the rows (xy) and (zw), we see that points of P^3 correspond to

FIGURE 10



non-zero 2×2 matrices X, considered up to a scalar factor. In terms of matrices the equation of H is $\det X = 0$; the equation of the plane P^2 is $\mathrm{Tr}\, X = 0$; and the equation of their intersection C is $X^2 = 0$. There are two families of lines on H, say A and B, and every tangent plane to H intersects H in the union of an A-line and a B-line. For every point on C, represented by a (nilpotent) matrix X_0, the equation of the plane tangent to H at this point is $\mathrm{Tr}\, XX_0 = 0$. Since $\mathrm{Tr} X_0 = 0$, scalar matrices satisfy this equation. Thus all planes tangent to H at points of C pass through the point $E \in P^3$ represented by scalar matrices. Now we can repeat the above proof: given a hexagon S in P^2 whose sides are tangent to C, construct a 3-dimensional hexagon S', using in turn A- and B-lines, such that S is the projection of S' from E onto P^2. (In the real case considered above, E was the point at infinity in the direction of the axis of the hyperboloid. Note that the assumption char $k \neq 2$ implies that the point E is not on P^2.) The main diagonals of S' meet at one point, since the three planes p_{14}, p_{25}, and p_{36} through the opposite sides of S' meet at one point. It follows that the main diagonals of S also meet at one point.

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Soliloquy

John B. Thoo

To use a calculator, or not to use a calculator, that is the question:
Whether 'tis nobler in the mind to suffer
The slings and arrows of outrageous paper and pencil computations,
Or to take arms against a sea of troubles,
And by opposing, end them. To long multiply, to long divide—
No more, and by using a calculator to say we end
The heart-ache and the thousand natural shocks
That paper and pencil computations are heir to, 'tis a consummation
Devoutly to be wish'd.

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